Ellipse

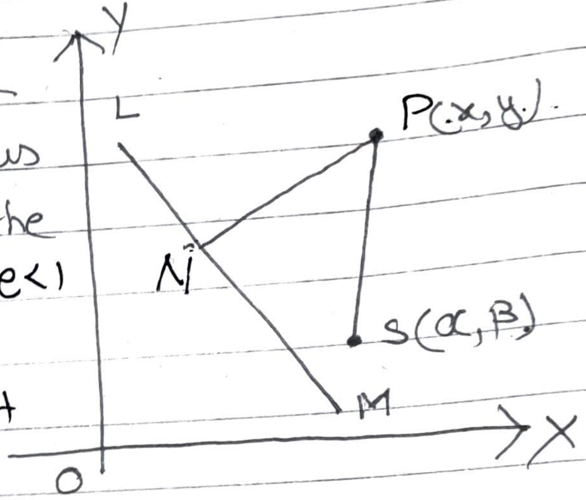
The conic section where eccentricity or the constant ratio between the point on curve and fixed line (the directrix) is less than '1'.

i.e. $e < 1$

is called ellipse.

General Eqn of ellipse

To find the equation of an ellipse whose focus is $S(\alpha, \beta)$ & directrix the line $ax + by + c = 0$ and $e < 1$



Let $P(x, y)$ be any point on the ellipse. Draw $PN \perp LM$ where eqn of LM is $ax + by + c = 0$ then by definition.

$$\frac{SP}{PN} = e \Rightarrow SP = e\,PN$$

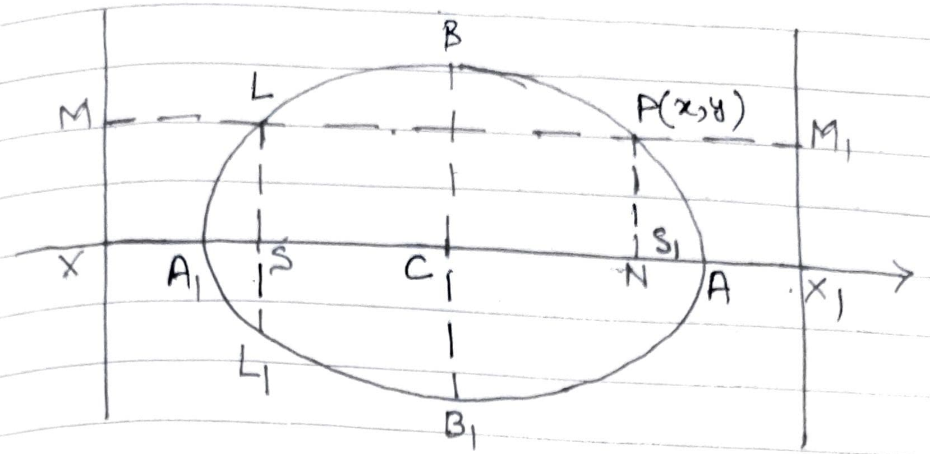
$$\Rightarrow SP^2 = e^2 PN^2$$

$$\Rightarrow (x-\alpha)^2 + (y-\beta)^2 = e^2 \left\{ \frac{(ax+by+c)^2}{a^2+b^2} \right\}$$

$$\Rightarrow \left\{ (x-\alpha)^2 + (y-\beta)^2 \right\}(a^2+b^2) = e^2 (ax+by+c)^2$$

which the required Equation.

Standard Equation of Ellipse



Let 'S' be the focus and MX be the directrix. Draw $SX \perp MX$. Divide SX internally at A_1 and externally at A in the ratio e:1 such that $\frac{A_1S}{A_1X} = e \Rightarrow A_1S = e\,A_1X$ —— (1)

and $\frac{AS}{AX} = e \Rightarrow AS = e\,AX$ —— (2)

By defn 'A' and 'A_1' lie on the ellipse.
So let $AA_1 = 2a$ and let 'C' be the middle point of AA_1 such that $A_1C = CA = a$.
From eqn (1) & (2) we get

$$A_1S + AS = e(A_1X + AX)$$
$$\Rightarrow AA_1 = e(CX - \cancel{CA_1} + \cancel{CA} + CX)$$
$$\Rightarrow AA_1 = 2e\,CX \Rightarrow \cancel{2}a = \cancel{2}e\,CX$$
$$\Rightarrow CX = a/e.$$

Again. $AS - A_1S = e(AX - A_1X)$

$$\Rightarrow (\cancel{CA} + CS) - (\cancel{CA_1} - CS) = e\,AA_1$$
$$\Rightarrow \cancel{2}CS = e \times (\cancel{2}a) \Rightarrow CS = ae.$$

Now take 'C' as origin and CA as x-axis and a line through $C \perp CA$ as y-axis

Let $P(x, y)$ be any point on the ellipse
Draw $PM \perp$ to the directrix
and $PN \perp CA$.
By definition $SP = e\,PM$

$$\Rightarrow SP^2 = e^2 PM^2$$

$$\Rightarrow SN^2 + NP^2 = e^2 (CN + CX)^2$$

$$\Rightarrow (x+ae)^2 + y^2 = e^2\left(x + \frac{a}{e}\right)^2$$

$$= \frac{\cancel{e^2}}{\cancel{e^2}}(ex+a)^2$$

$$\Rightarrow x^2(1-e^2) + y^2 = a^2(1-e^2)$$

$$\Rightarrow \frac{x^2\cancel{(1-e^2)}}{a^2\cancel{(1-e^2)}} + \frac{y^2}{a^2(1-e^2)} = 1$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$$

Put $b^2 = a^2(1-e^2)$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which the required Equation.

Ques Find the equation of ellipse of eccentricity $\frac{1}{2}$, focus $(1, -1)$ and directrix $x - y = 3$.

Soln Eqn of ellipse is given by.

$$(x-\alpha)^2 + (y-\beta)^2 = e^2\frac{(ax+by+c)^2}{(a^2+b^2)}$$

Here $\alpha = 1, \beta = -1, e = \frac{1}{2}, a = 1, b = -1, c = -3.$

Substituting these values we get

$$(x-1)^2+(y+1)^2=\frac{1}{4}\cdot\frac{(x-y-3)^2}{2}$$

$$\Rightarrow 8\{(x-1)^2+(y+1)^2\}=(x-y-3)^2.$$

is the required Equation.

Ques 4 The foci of an ellipse are $(\pm 2, 0)$ and eccentricity $\frac{1}{2}$. Find its equation.

Since $ae=2$. and $b^2=a^2\left(1-\frac{1}{4}\right)$ $\because e=\frac{1}{2}$

$\Rightarrow a\cdot\frac{1}{2}=2 \Rightarrow a=4$

$$b^2=4^2\left(1-\frac{1}{4}\right)=\cancel{16}^{4}\cdot\frac{3}{\cancel{4}}=12$$

Eqn. of ellipse is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\Rightarrow \frac{x^2}{16}+\frac{y^2}{12}=1.$$

Ans.